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ISOTHERM MIGRATION METHOD IN TWO DIMENSIONS

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Abstract—The Isotherm Migration Method is extended to two dimensions. The equations are formulated and a convenient finite-difference method of solution is described for a variety of initial and boundary conditions. Particular attention is devoted to Stefan problems in which phase changes occur on a moving interface. As an example the solidification of a square prism of fluid is solved in detail and the numerical results are compared with those obtained by earlier authors.

1. NOMENCLATURE

u, temperature;
x, y, cartesian coordinates;
t, time;
$f(x, y, t)$, $= 0$, interface contour;
β, a physical constant;
n, direction of normal to the interface;
v_n, normal velocity of the interface;
u_r, rth isotherm having a temperature u_r;
δu, temperature step;
δt, time step;
$\delta x, \delta y$, mesh sizes in x and y directions respectively;
b, radius of circle;
$y^k_{i,j}$, value of y for the ith isotherm at $x = j\delta x$ at $t = k\delta t$.

2. INTRODUCTION

IN THEIR most familiar forms the heat-conduction equation and its solutions express temperature as a function of the space coordinates and time. An alternative is to regard the temperature as an independent variable and one of the space coordinates then becomes a dependent variable. Chernous'ko [1] and Dix and Cizek [2] have explored this idea for problems in one space variable, x. Instead of writing temperature $u = u(x, t)$ where t is time, they propose to express x as a function of u and t, i.e. $x = x(u, t)$. By some numerical process or otherwise we then calculate the positions of given temperatures at different times. Essentially we track the movement of isotherms and the name Isotherm Migration Method (IMM) is appropriate. Crank and Phale [3] have used the method to solve the problem of the melting of a plane sheet of ice.

In the present paper we extend the IMM to two space dimensions. One convenient numerical method of evaluating solutions of the transformed equation is described. Some of the advantages of the IMM are mentioned and also some of the difficulties.

As an example, the method is used to study the problem posed by the solidification of a square prism of fluid initially at a constant temperature throughout. Numerical values are compared with some earlier results obtained by other methods.

3. TRANSFORMED EQUATIONS

The heat-conduction equation in two space dimensions x, y is usually written in non dimensional terms

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{1}$$

Since the temperature u is constant along an isotherm we have

$$\mathrm{d}u = \left(\frac{\partial u}{\partial y}\right)_{x,t} \mathrm{d}y + \left(\frac{\partial u}{\partial t}\right)_{x,y} \mathrm{d}t = 0 \tag{2}$$

and so

$$\left(\frac{\partial y}{\partial t}\right)_{u,x} = -\left(\frac{\partial u}{\partial t}\right)_{x,y} \Big/ \left(\frac{\partial u}{\partial y}\right)_{x,t} = -\left(\frac{\partial u}{\partial t}\right)_{x,y} \left(\frac{\partial y}{\partial u}\right)_{x,t} \tag{3}$$

Substituting (1) in (3) and dropping suffices we obtain

$$\frac{\partial y}{\partial t} = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\left(\frac{\partial y}{\partial u}\right) \tag{4}$$

and remembering that

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial y}{\partial u}\right)^{-1} = -\frac{\partial^2 y}{\partial u^2}\left(\frac{\partial y}{\partial u}\right)^{-3} \tag{5}$$

we find

$$\frac{\partial y}{\partial t} = -\left\{\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 y}{\partial u^2}\left(\frac{\partial y}{\partial u}\right)^{-3}\right\}\left(\frac{\partial y}{\partial u}\right). \tag{6}$$

This equation represents y as a function of u, x and t.

In order to illustrate a numerical method of solution we consider initial and boundary conditions specified for a square region by

$$u = 0, \quad y = 0, \quad 0 \leqslant x \leqslant 1, \quad t > 0, \tag{7}$$

$$u = 1, \quad y = 1, \quad 0 \leqslant x \leqslant 1, \quad t > 0, \tag{8}$$

$$u = y^2, \quad x = 0, \quad 0 \leqslant y \leqslant 1, \quad t > 0, \tag{9}$$

$$u = y, \quad x = 1, \quad 0 \leqslant y \leqslant 1, \quad t > 0, \tag{10}$$

$$u = \tfrac{1}{2}(x+y), \quad 0 < x < 1, \quad 0 < y < 1, \quad t = 0. \tag{11}$$

We work on a $u-x$ grid choosing δu and δx such that $u_i = u_0 + i\delta u$, $i = 1, 2, \ldots, N$ and $x_j = x_0 + j\delta x$, $j = 1, 2, \ldots, M$. Conditions (7) and (8) imply $y = 0$ for all points x_j on the $u_0 = 0$ grid line and $y = 1$ for all points x_j on the $u_N = 1$ grid line respectively for all $t > 0$. On the $x = x_0 = 0$ line we have for $t > 0$, $y = +u_i^{\frac{1}{2}}$ for all points $i(i = 1, 2, \ldots, N)$ and on the line $x = x_M = 1$ for $t > 0$ (10) gives $y = u_i$ for all points $i(i = 1, 2, \ldots, N)$. Initially, when $t = 0$, values of y are obtained at all internal grid points using (11), i.e. at the point (u_i, x_j) $y_{ij} = 2u_i - x_j$. Thus the positions of the isotherms are known everywhere in the square region at $t = 0$ and values of y are specified on all four boundaries of the $u-x$ grid for all time t. We note that $u_0 + r\delta u$ is the rth isotherm on which the temperature is u_r in the $x-y$ plane.

We now calculate the values of y on this grid at successive time steps δt. Suppose the numerical solution has proceeded as far as $t = k\delta t$ so that values of y are known at this time at all points on the $u-x$ grid. Let $y_{i,j}^k$ represent the value of y for the ith isotherm at $x = j\delta x$ at time $k\delta t$. Then the corresponding value $y_{i,j}^{k+1}$ at the next time step $(k+1)\delta t$ can be obtained explicitly by the following finite-difference replacement of (6),

$$\frac{y_{i,j}^{k+1} - y_{i,j}^k}{\delta t} = -\left(\frac{y_{i,j}^k - y_{i-1,j}^k}{\delta u}\right)\frac{\partial^2 u}{\partial x^2} - 4\left\{\frac{y_{i-1,j}^k - 2y_{i,j}^k + y_{i+1,j}^k}{(y_{i+1,j}^k - y_{i-1,j}^k)^2}\right\} \tag{12}$$

with $i = 1, 2, \ldots, N-1$ and $j = 1, 2, \ldots, M-1$.

In order to evaluate $\partial^2 u/\partial x^2$ appearing in the right side of (12) we interpolate (or extrapolate) linearly the values of u corresponding to $y_{i,j}$ at x_{j-1} and also at x_{j+1}. A typical formula for calculating u at $x = x_{j-1}$ is

$$u = \frac{u_{i+1}(y_{i,j-1} - y_{i,j}) - u_i(y_{i+1,j-1} - y_{i,j})}{y_{i,j-1} - y_{i+1,j-1}}. \tag{13}$$

It is not the intention in this paper to proceed with a detailed numerical evaluation of the solution of this problem. Its purpose is simply to introduce the IMM method and to facilitate a quantitative description of some advantages and difficulties ot the method.

First we observe that the transformed equation (6) is non-linear even for constant thermal properties when the original equation (1) is linear. This is relatively unimportant if an explicit difference scheme is used as in (12) but would call for the solution of a set of non-linear algebraic equations if an implicit difference replacement were introduced. On the other hand the method can offer appreciable advantages for problems involving variable heat parameters particularly when they are temperature dependent. The parameters need not be evaluated for the different set of temperatures calculated at each time step at the points of an $x-y$ grid as in the traditional finite-difference solution of equation (1). Instead the parameters are evaluated once and for all before the IMM starts and only for the constant u lines of the $u-x$ grid.

The example used above avoids two difficulties which can beset the use of the IMM. In the initial temperature and in the boundary conditions y is a single-valued function of u and x; that is there is only one value of y for any given point on the $u-x$ grid. If, for example, the condition (9) were to be replaced by

$$u = 0, \quad x = 0, \quad 0 \leqslant y \leqslant 1, \quad t > 0 \tag{14}$$

then y would be multi valued at the point $x = 0, u = 0$ on the $u-x$ grid and could take any value in the range $0 \leqslant y \leqslant 1$. There would never be any values for y on the boundary $x = 0$, $0 < u < 1$. It is still possible, however, to apply the formula (12) at points on the grid for which $x = \delta x$, i.e. $j = 1$. The only term which involves the line $j = 0$ is $\partial^2 u/\partial x^2$ which at a typical point $(i, 1)$ is replaced by the usual 3-point formula containing values of u at $x = 0$, δx, $2\delta x$ for which $y = y_{i,1}$. According to (14) at $x = 0$, $u = 0$ for all values of y in the range $0 < y < 1$ and hence for $y = y_{i,1}$. At $x = \delta x$, we insert $u = u_{i,1}$ and for $x = 2\delta x$ we use an interpolated or extrapolated value of u based on (13).

Other boundary conditions on $x = 0$ can be dealt with by a parabolic or other extrapolation procedure based on the two internal isotherms nearer the boundary. An example is given in Section 5 below.

We note that a condition on the boundary $y = 0$ such as $u = 0$, $0 < x < 1$, $y = 0$, presents no difficulty. On the $u-x$ grid we simply insert $y = 0$ at each grid point on the line $u = 0$ for $0 < x < 1$.

The decision as to whether the IMM transformation should be applied to $\partial^2 u/\partial y^2$ as above or to $\partial^2 u/\partial x^2$ may be influenced by the nature of the boundary conditions along $x = 0$ and $y = 0$. If the initial condition (11) were to be replaced by

$$u = 0, \quad 0 < x < 1, \quad 0 < y < 1, \quad t = 0, \tag{15}$$

the value of y at any point on the grid line $u = 0$ other than $x = 1$ could be anywhere in the range $0 \leqslant y < 1$ at $t = 0$. No isotherms other than $u = 0$ exist at $t = 0$ and no initial values of y are available at any internal point on the $u-x$ grid. In such a situation, as in the corresponding one-dimensional case described by Dix and Cizek [2], we must generate an initial set of isotherms. An analytic or some alternative solution is needed to provide the temperature distribution at some small time t_0 from which the IMM can proceed. It is not unusual to have to use some special procedure for starting a numerical solution, particularly if singularities exist at $t = 0$. Sometimes a few time steps can be evaluated using a finite-difference form of the original equation (1) on an $x-y$ grid and then by suitable interpolation y values can be transferred to a $u-x$ grid and the solution continued by the IMM.

The temperature u is always a single valued function of x and y both initially and at any subsequent time. The temperature at any point can only have one value at a given time. We have seen however that y can be a multi valued function of u both at $t = 0$ or on the boundaries when $t > 0$. In some problems y may be multi-valued as the solution proceeds anywhere in the domain of interest.

If the temperature on one boundary first rises and later falls it is possible that the same temperature will occur at two different points in the region at one time. Another example is provided by a square region initially at zero temperature and of which the boundaries are subsequently maintained at a constant non-zero temperature. Because of the symmetry of the isotherms, for any given x there will be two values of y

at which the temperature u is the same. In the example discussed below in Section 5 use is made of the symmetry to avoid the occurrence of double valued functions. Where symmetry does not exist, or where to make use of it might introduce fresh difficulties, other convenient methods of handling the double values of y need to be explored. Such methods lie outside the scope of the present paper. Instead we turn to another variant of a boundary condition for which IMM is particularly suitable.

4. MOVING BOUNDARIES

In many problems of practical importance one or more conditions are specified on boundaries which move through the medium. They include problems of melting and freezing in which a moving interface separates the liquid from the solid phase. The temperature on the interface remains constant at the melting temperature. The velocity of the interface is determined by the rate at which the heat required for the phase change is supplied and removed by conduction. A second boundary condition expresses this heat balance at the interface. Thus the solution of the partial differential equation of heat flow is coupled with the unknown motion of the interface. These problems are classed as Stefan problems and different investigators have tried a variety of methods to obtain solutions [4]. Most of the attempts have confined attention to problems in one space dimension though Allen and Severn [5] applied relaxation techniques to two dimensional problems and Poots [6] obtained approximate analytic solutions. More recently Lazaridis [7] formulated general equations for a solidification problem in three dimensions and obtained solutions by finite-difference methods. The IMM has an obvious attraction for the solution of Stefan problems in which the temperature is constant on a moving boundary. The IMM is essentially concerned with the tracking of isotherms through a medium and the phase-change boundary is itself an isothermal surface. Thus no special problems arise in calculating its motion except the necessary IMM transformation of the melting condition.

The IMM has been used successfully by Crank and Phale [3] to solve a one-dimensional problem of the melting of a plane sheet of ice. Rose [8] used a similar transformation.

We now formulate the condition on a melting interface in two dimensions in a form suitable for IMM. We denote by $f(x, y, t) = 0$ the liquid/solid interface at time t. The net rate at which heat becomes available at the interface is given by $K_S(\partial u_S/\partial n) - K_L(\partial u_L/\partial n)$ where u_L and u_S denote temperatures in the liquid and solid phases respectively, K_L and K_S are the corresponding heat conductivities and n is measured along the normal to the interface from liquid to solid. If the velocity of the interface is v_n along the normal, n, the necessary rate of supply of the heat of transformation L is $L\rho v_n$ where ρ is the density assumed the same for solid and liquid phases. To secure a heat balance we must have

$$k_S \frac{\partial u_S}{\partial n} - k_L \frac{\partial u_L}{\partial n} = L\rho v_n. \tag{16}$$

Patel [9] showed that (16) can be written in a more convenient form for our present purpose which is to evaluate $\partial y/\partial t$ for points on the interface $f(x, y, t) = 0$. The modified form of (16) in present nomenclature is

$$\frac{\partial y}{\partial t} = \frac{1}{\beta}\left\{1 + \left(\frac{\partial y}{\partial x}\right)^2\right\}\left\{k_S\left(\frac{\partial y}{\partial u_S}\right)^{-1} - k_L\left(\frac{\partial y}{\partial u_L}\right)^{-1}\right\}, \tag{17}$$

where we have written $\partial u/\partial y = (\partial y/\partial u)^{-1}$.

On the interface (17) replaces (6) which holds at all other points of the region. Lazaridis [7] uses Patel's form of interface condition in three dimensions and comments that its usefulness lies in the fact that (17), for example, is quasi one-dimensional, sinze the temperature gradient is taken in one dimension only.

5. SOLIDIFICATION OF A SQUARE PRISM OF FLUID

As an example of the application of the IMM to a two-dimensional Stefan problem we consider the following.

An infinitely long prism is initially filled with a fluid at the fusion temperature $u = 1$. The temperature on its surface is subsequently maintained constant at $u = 0$ below the fusion temperature so that solidification proceeds from the surface inwards. Let us assume that the prism extends between $-1 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant 1$.

Formulated in non-dimensional terms and denoting temperature in the solid phase by u, we require a solution of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{18}$$

subject to the boundary conditions

$$u = 0 \text{ on } g(x, y) \equiv (x^2 - 1)(y^2 - 1) = 0, \quad t \geqslant 0, \tag{19}$$

with $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$ and

$$u = 1 \text{ on } f(x, y, t) = 0, \quad t > 0, \tag{20}$$

where $f(x, y, t) = 0$ is the liquid/solid interface at time t. Initially we have

$$f(x, y, 0) \equiv g(x, y) = 0, \quad t = 0, \tag{21}$$

because the solidification process has not yet commenced.

In addition to (20), the second condition to be satisfied on the moving interface is the appropriate form of (16) namely

$$\frac{\partial u}{\partial n} = -\beta v_n \quad \text{on} \quad f(x, y, t) = 0, \tag{22}$$

where n is the outward normal to $f(x, y, t) = 0$, v_n is the velocity of the interface in the direction of n and β is a constant depending on the thermal properties of the material undergoing the phase change. The condition (22) is simpler than the more general condition (16) because in this example the liquid phase is always at the uniform temperature $u = 1$ and so there is no temperature gradient in the liquid phase.

In this example the problem is symmetrical about the axes. We need consider only one quadrant of the prism enclosed say between the sides $x = 1$, $y = 1$ and the axes. Furthermore, the symmetry implies a zero flux of heat along the axes so that the following boundary conditions have to be satisfied

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} \quad \text{and} \quad \left(\frac{\partial u}{\partial y}\right)_{y=0} = 0. \tag{23}$$

The IMM form of (18) is equation (6) and that of (22) is easily seen to be

$$\frac{\partial y}{\partial t} = \frac{1}{\beta}\left\{1+\left(\frac{\partial y}{\partial x}\right)^2\right\}\left(\frac{\partial y}{\partial u}\right)^{-1} \tag{24}$$

from (17).

The structure and properties of an isotropic medium in the neighbourhood of any point are the same in all directions through the point. Because of this symmetry at a point on an isothermal surface the flux vector of heat flow must be normal to that surface. By a similar appeal to symmetry it follows that every point on the isotherm moves in the direction of the normal to it.

By making use of this property we can deduce (24) by a somewhat shorter argument than that of Patel [9]. Thus differentiating (20) with respect to t we get

$$\frac{\partial u}{\partial t} = -\left\{\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}\right\}$$
$$= -\left(\mathbf{i}\frac{\partial u}{\partial x}+\mathbf{j}\frac{\partial u}{\partial y}\right)\cdot\left(\mathbf{i}\frac{\partial x}{\partial t}+\mathbf{j}\frac{\partial y}{\partial t}\right) \tag{25}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x and y directions respectively.

If $P(x, y)$ is a point on the interface $u(x, y, t) = 1$, the first term on the right side of (25) is the gradient of u at P, and the second term gives the velocity of P. Furthermore, accepting that every point on the isotherm (interface) moves along the normal to it, we write (25) as

$$\frac{\partial u}{\partial t} = -(\nabla u)\,.\,(-v_n\mathbf{n}),$$
$$= -\frac{1}{\beta}(\nabla u)\,.\,(\nabla u) \tag{26}$$

from (22), where $\mathbf{n}$ is the outward normal to $f(x, y, t) = 0$. But we have

$$\left(\frac{\partial y}{\partial t}\right)_{u,x} = -\left(\frac{\partial u}{\partial t}\right)_{x,y}\left(\frac{\partial y}{\partial u}\right)_{x,t} \tag{27}$$

and combining (26) and (27) gives

$$\frac{\partial y}{\partial t} = \frac{1}{\beta}\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right\}\frac{\partial y}{\partial u} \tag{28}$$

from which we immediately regain (24).

We evaluate a numerical solution on a $u-x$ grid as in Section 3, choosing δu and δx such that $u_i = u_0 + i\delta u$, $i = 1, 2, \ldots, N$ $(u_0 = 0,\ u_N = 1)$ and $x_j = x_0 + j\delta x$, $j = 1, 2, \ldots, M$ $(x_0 = 0, x_M = 1)$. We calculate the values of y on this grid at successive time-steps δt. A convenient finite-difference form of (24) is

$$\frac{y_{N,j}^{k+1}-y_{N,j}^k}{\delta t} = \frac{1}{\beta}\left\{1+\left(\frac{y_{N,j}^k-y_{N,j-1}^k}{\delta x}\right)^2\right\}\frac{\delta u}{y_{N,j}^k-y_{N-1,j}^k}, \tag{29}$$

giving $y_{N,j}^{k+1}$ explicitly for $j = 1, 2, \ldots$.

This uses a first order, backward difference approximation to $(\partial u/\partial y)^{-1}$ which is convenient at the interface. To be consistent we have replaced (18) by

$$\frac{y_{i,j}^{k+1}-y_{i,j}^k}{\delta t} = -\left(\frac{y_{i,j}^k-y_{i-1,j}^k}{\delta u}\right)\frac{\partial^2 u}{\partial x^2} - \frac{y_{i-1,j}^k-2y_{i,j}^k+y_{i+1,j}^k}{(y_{i,j}^k-y_{i-1,j})^2} \tag{30}$$

rather than by (12) though the central difference form would be possible. The method of treating $\partial^2u/\partial x^2$ described in Section 3 is used here based on formula (13).

On the y-axis (6) breaks down because of the second of (23). Therefore we fit a parabola through (x_1, y_1) and (x_2, y_2) to get the value of y at $x = 0$, remembering that $\partial y/\partial x = 0$ on the y-axis. At the other end of the isotherm, i.e. away from $x = 0$, we fit a circle passing through the last two computed values of y, to give its value at the next grid point in the x direction. Since the isotherm is symmetrical about the diagonal $y = x$ the equation of the circle may be written as

$$(x-a)^2+(y-b)^2 = b^2, \tag{31}$$

where a and b are unknowns and the centre of the circle is at the point (a, a) on $y = x$. In order to describe the detailed process at this end we refer to Fig. 1 which shows the positions of an isotherm in the $x-y$ plane at times t and $t+\delta t$.

We compute those values of y at $(k+1)\delta t$ from (30) for which $y(x) > x+\delta x$ at time $k\delta t$. If $y(x) \leqslant x+\delta x$, the process is shifted one step back towards the y-axis. Let us suppose that in Fig. 1 the values of y corresponding to $x = x_i$, $i = 0, 1, \ldots, (r+1)$ are known at time $k\delta t$ and are given by y_i^k. It should be remembered that the point $R(y_{r+1}^k)$ has been projected by fitting a circle (31) through $P(y_{r-1}^k)$ and $Q(y_r^k)$. When we find that at time $(k+1)\delta t$, $y_r^{k+1} \leqslant x_r+\delta x$ we calculate y_i^{k+1} using (30) for $i = 1, 2, \ldots, (r-1)$ only and $W(y_r^{k+1})$ is then obtained by fitting the circle (31) through $S(y_{r-2}^{k+1})$ and $T(y_{r-1}^{k+1})$. By choosing $y(x) > x+\delta x$ we

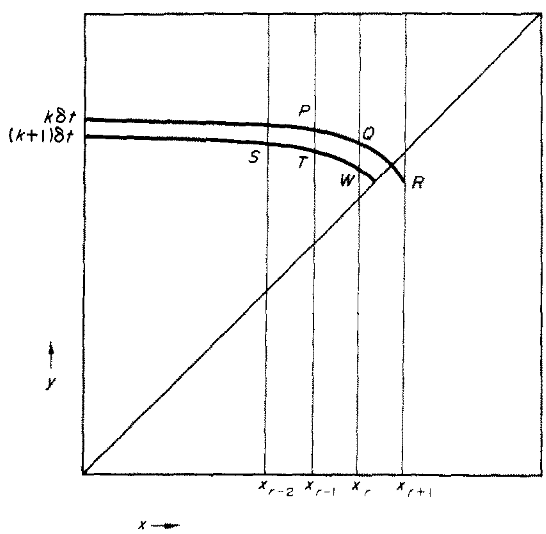

FIG. 1. Positions of an isotherm at times t and $t+\delta t$.

make certain that the circle fitted through the two neighbouring points will cut the next grid line parallel to the y-axis because of the symmetry about the diagonal $y = x$.

We have chosen to take advantage of the symmetry in this example partly in order to reduce the number of points at which the difference equations (29) and (30) have to be evaluated and partly to avoid double-valued functions. Any saving in computer time is offset to some extent by the interpolation and extrapolation procedures used on the axes and the line $y = x$ though these proved very convenient in practice. No attempt is made in this paper to study the stability of the method nor to carry out a formal error analysis. These must be the subject of further investigation.

6. RESULTS AND DISCUSSION

In order to start the IMM we must first calculate the positions of some isotherms and the interface after solidification has been taking place for a short time. We use the one-parameter integral method of Poots [6] to calculate isotherm positions at $t = 0{\cdot}0461$. The mesh sizes are $\delta u = \delta x = 0{\cdot}1$ and the time step $\delta t = 0{\cdot}0001$. The value of β is $\beta = 1{\cdot}561$, the same as that used by Lazaridis [7] and earlier workers. Table 1 gives values of y on the solid/liquid interface for fixed values of x at various times. As there is symmetry about the diagonal $y = x$, only the values of y which are above the diagonal are tabulated. In cases where $y(x) > x + \delta x$ at the grid point near the diagonal one more value of y corresponding to the next grid value of x is also given.

Interface contours are plotted in one quarter of the prism in Fig. 2 corresponding to the values in Table 1. The positions of the various isotherms are shown in Fig. 3 at $t = 0{\cdot}495$ when the computations are stopped because there are only two values of y left corresponding to the grid points in the x direction on the solid–liquid interface where $u = u_N = 1$.

The graphs in Fig. 4 and 5 show the proportion of the solidified matter along the axes and the diagonal respectively against time. Corresponding points obtained by the numerical methods of Lazaridis [7] and Allen and Severn [5] are also plotted for comparison.

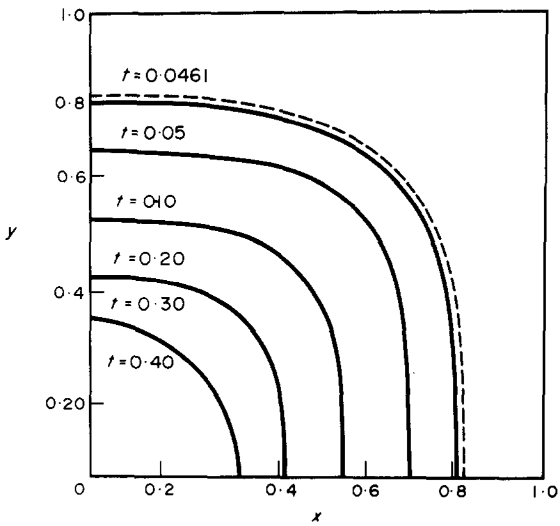

FIG. 2. Positions of the interface at $t = 0{\cdot}05(0{\cdot}05)0{\cdot}45$. The dotted line shows the position at $t = 0{\cdot}0461$ obtained from the Poot's one-parameter integral method.

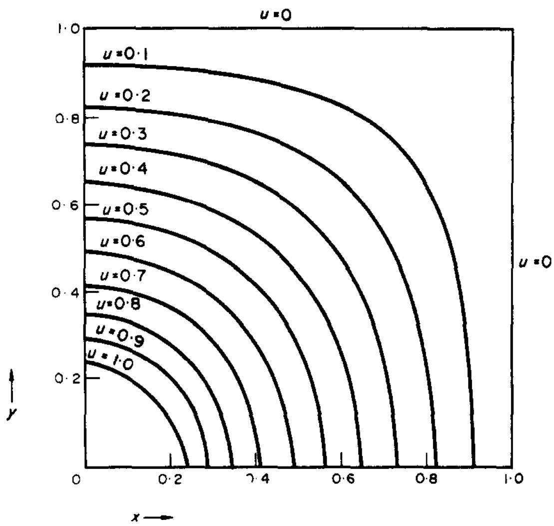

FIG. 3. Positions of the isotherms having temperatures $u = 0{\cdot}0(0{\cdot}1)1{\cdot}0$ at $t = 0{\cdot}495$.

Table 1. Values of the y coordinate on the solid–liquid interface for fixed values of x at various times. Solution starts from the values taken from the Poots one-parameter method at $t = 0{\cdot}0461$

	x						
t	0·0	0·1	0·2	0·3	0·4	0·5	0·6
0·05	0·8125	0·8106	0·8048	0·7940	0·7764	0·7476	0·6904
0·10	0·6979	0·6965	0·6921	0·6836	0·6683	0·6392	0·5606
0·15	0·6157	0·6141	0·6095	0·6000	0·5810	0·5201	
0·20	0·5473	0·5453	0·5394	0·5268	0·4789		
0·25	0·4865	0·4838	0·4755	0·4567	0·3894		
0·30	0·4302	0·4263	0·4146	0·3654			
0·35	0·3766	0·3708	0·3534	0·2859			
0·40	0·3337	0·3158	0·2623				
0·45	0·2816	0·2585	0·1893				
0·495	0·2376	0·2056	0·1097				

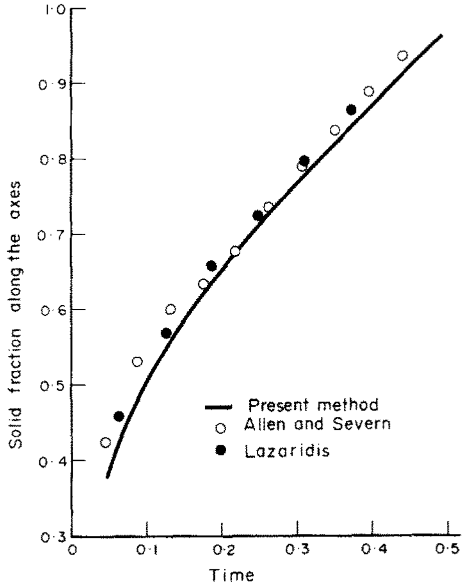

FIG. 4. Graph of the solid fraction along the axes against time.

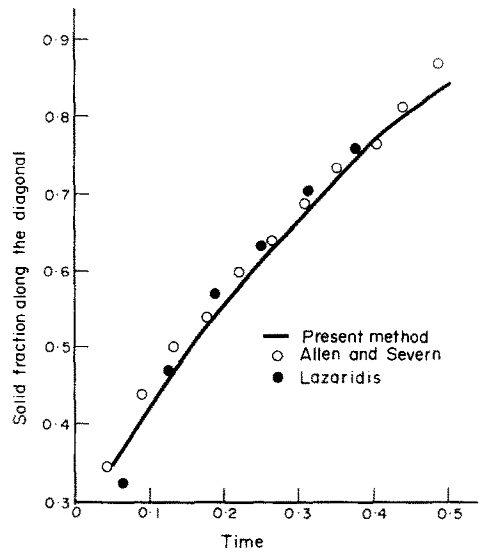

FIG. 5. Graph of the solid fraction along the diagonal of the prism against time.

The results obtained by the IMM agree reasonably well with the earlier values. Some of the discrepancy may be due to the use of the Poots [6] approximate method to start the IMM solution.

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METHODE DE MIGRATION ISOTHERME A DEUX DIMENSIONS

Résumé—La méthode de migration isotherme est étendue à deux dimensions. Les équations une fois formulées, une méthode de résolution appropriée basée sur un schéma de différences finies est présentée pour diverses conditions aux limites et initiales. Une attention particulière est donnée aux problèmes de Stefan dans lesquels des changements de phase interviennent sur un interface en mouvement. A titre d'exemple le problème de solidification d'un priseme carré de fluide est résolu en détail et les résultats numériques sont comparés à ceux obtenues par des auteurs antérieurs.

ZWEIDIMENSIONALE, ISOTHERME MIGRATIONSMETHODE

Zusammenfassung—Die isotherme Migrationsmethode wird auf zweidimensionale Fälle erweitert. Die hergeleiteten Gleichungen werden mit einer geeigneten finite-Differenzen-Methode für verschiedene Anfangs- und Randbedingungen gelöst. Besonderes Augenmerk wird dem Stefan-Problem gewidmet, bei dem Phasenänderungen in einer bewegten Grenzfläche auftreten. Am Beispiel eines in Quadratprismenform erstarrenden Fluids erfolgt eine ausführliche Beschreibung des Lösungswegs und ein Vergleich der Ergebnisse mit denen aus früheren Veröffentlichungen.

ДВУМЕРНЫЙ МЕТОД ИЗОТЕРМИЧЕСКОЙ МИГРАЦИИ

Аннотация — Метод изотермической миграции распространяется на двумерную задачу. Сформулированы уравнения и описан метод конечных разностей для их решения для целого ряда начальных и граничных условий. Особое внимание обращается на задачу Стефана, когда изменение фазы происходит на движущейся границе раздела. В качестве примера приводится подробное решение для процесса затвердевания жидкости в форме квадратной призмы. Численные результаты сравниваются с более ранними результатами, полученными другими исследователями.